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On sets of points that determine only acute angles

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ABSTRACT

We show that for every integer d there is a set of points in $\mathbb{E}^d$ of size $\Omega((\frac{2}{\sqrt{3}})^d\sqrt{d})$ such that every angle determined by three points in the set is smaller than $\pi/2$. This improves the best known lower bound by a $\Theta(\sqrt{d})$ factor.

1. Introduction

Let P be a set of points in $\mathbb{E}^d$. Given three points $a, b, c \in P$ the angle $\angle abc$ is *acute* if it is smaller than $\pi/2$, *right* if it is equal to $\pi/2$, and *obtuse* if it is greater than $\pi/2$. Clearly, the 2^d vertices of a d-dimensional cube (or box) determine no obtuse angle. Erdős conjectured, and Danzer and Grünbaum [5] proved that this is best possible, that is, any set of $2^d + 1$ points in $\mathbb{E}^d$ has three points that determine an obtuse angle.

Denote by $f(d)$ the maximum size of a set of points in $\mathbb{E}^d$ that determine only acute angles. Hence $f(d) \leq 2^d$. Danzer and Grünbaum [5] showed that $f(d) \geq 2d - 1$ and conjectured that this bound is tight. However, Erdős and Füredi [7] (see also [1,2]) gave a proof from THE BOOK that this conjecture is false. They used a probabilistic argument to show that $f(d) \geq \lfloor \frac{1}{2}(\frac{2}{\sqrt{3}})^d \rfloor \approx 0.5 \times 1.1547\ldots^d$. They also claimed, but gave no proof or hint, that a more complicated proof yields the better bound $f(d) \geq (\sqrt[4]{2} - o(1))^d \approx 1.189\ldots^d$. Recently, Bevan [4] improved the former bound by a constant factor to $2\lfloor \frac{1}{3}(\frac{2}{\sqrt{3}})^{d+1} \rfloor \approx 0.77 \times 1.1547\ldots^d$. Here we show that by using a method of Komlós et al. [8] this bound can be improved by a $\Theta(\sqrt{d})$ factor. It is quite likely that the claimed bound, if true, can also be improved by this method.

2. The lower bound

In this section we prove

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Theorem 1. $f(d) = \Omega\left(\left(\frac{2}{\sqrt{3}}\right)^d \sqrt{d}\right)$.

As in the proof of Erdős and Füredi [7], we pick at random points from the d-dimensional hypercube. It then follows that the number of triplets of points that form a right angle (recall that the hypercube has no obtuse angles) is small. Instead of removing the points involved in a right angle, as in the proof of Erdős and Füredi [7] and its refinement [4], we show that there are not too many subsets of four points that determine two right angles. To get the better lower bound we then use the following theorem of Bertram-Kretzberg and Lefmann [3], which is a special case of a theorem of Duke et al. [6] that, in turn, extends a theorem of Komlós et al. [8].

Theorem 2 (*Special Case of Corollary 2.7 [3]*). *Let H be a 3-uniform hypergraph with n vertices and average degree at most t^2 where $t \to \infty$ with $n \to \infty$. If the number of 2-cycles in H is at most $c \cdot n \cdot t^{3-\epsilon}$, for some constants $c, \epsilon > 0$, then one can find in polynomial time an independent set of size $\Omega(n\sqrt{\log t}/t)$ in H.*

A 2-cycle in a 3-uniform hypergraph is formed by four vertices a, b, c, d and two hyperedges $\{a, b, c\}, \{a, b, d\}$.

We now describe the proof in detail. Pick n points $x_1, x_2, \ldots, x_n$ (n to be determined later) from the d-dimensional hypercube $\{0, 1\}^d$ in the following way: $x_{ij} = 0$ with probability $1/2$ and $x_{ij} = 1$ with probability $1/2$, $1 \le i \le n$, $1 \le j \le d$. Note that it might happen that $x_i = x_j$. Observe that if a triplet x_i, x_j, x_k forms a right angle at x_j then the dot product of the vectors $x_i - x_j$ and $x_k - x_j$ is zero. In other words, for every $1 \le \ell \le d$ we have $x_{i\ell} - x_{j\ell} = 0$ or $x_{k\ell} - x_{j\ell} = 0$. For a fixed ℓ this happens with probability $3/4$; thus the probability that x_i, x_j, x_k form a right angle at x_j is at most $(3/4)^d$. We say that a triplet x_i, x_j, x_k is *bad* if one of the dot products that it defines is zero. It follows that the expected number of bad triplets is $3\binom{n}{3}\left(\frac{3}{4}\right)^d$.

Next we define a 3-uniform hypergraph H whose vertices are the selected points and whose hyperedges are the bad triplets. We now wish to estimate the number of 2-cycles in H. A 2-cycle involves four vertices, two of which are the endpoints of the cycle, and two bad triplets. Let x_i, x_j, x_k, x_ℓ be four chosen vertices. For $d = 1$ the probability that x_i, x_j, x_k and x_j, x_k, x_ℓ both form a right angle (more precisely a dot product equals zero) is at most $5/8$. This can be verified by considering all the 2^4 possible values of x_i, x_j, x_k, x_ℓ and the possible choices of the apexes of the angles. Thus the expected number of 2-cycles is at most $\binom{4}{2}\binom{n}{4}\left(\frac{5}{8}\right)^d$.

Using Markov's inequality we get that with high probability, the numbers of hyperedges and 2-cycles in H are both not greater than their expected values by more than a constant factor. Thus we would like the following inequalities to hold:

$$t^2 > \frac{c_1 n^3 \left(\frac{3}{4}\right)^d}{n} \tag{1}$$

$$c_2 n^4 \left(\frac{5}{8}\right)^d < cnt^{3-\epsilon}. \tag{2}$$

By fixing, say, $n = c_3\left(\frac{2}{\sqrt{3}}\right)^{101d/100}$ and $t = c_4 n^{1/101} = c_5\left(\frac{2}{\sqrt{3}}\right)^{d/100}$, for suitable c_3, c_4, c_5, it is easy to verify that the inequalities hold. Therefore, H has an independent set of size $\Omega(n\sqrt{\log t}/t) = \Omega\left(\left(\frac{2}{\sqrt{3}}\right)^d \sqrt{d}\right)$. Clearly, this set does not contain bad triplets. Recall that the original random set might contain duplicated vertices; however, a triplet containing two identical vertices is bad, and thus the independent set contains no duplicated vertices.

Remark 1. Since we require $t = \Omega\left(n\left(\frac{\sqrt{3}}{2}\right)^d\right)$, the guaranteed size of the independent set is $O\left(\left(\frac{2}{\sqrt{3}}\right)^d \sqrt{\log t}\right)$. Thus, the base of the exponential part of the lower bound cannot be improved using this method.

Remark 2. Since Theorem 2 also guarantees a polynomial-time algorithm to find the independent set, one can find the guaranteed set of points that do not determine acute angles in time polynomial in the size of the set.

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